

1. z – TRANSFORM

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1.1 Definition: The z-transform $X(z)$ of a causal discrete – time signal $x(n)$ is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} \quad (4.1)$$

z is a complex variable of the transform domain and can be considered as the complex frequency. Remember index n can be time or space or some other thing, but is usually taken as time. As defined above, $X(z)$ is an integer power series of z^{-1} with corresponding $x(n)$ as coefficients. Let's expand $X(z)$:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \quad (4.2)$$

In general one writes

$$X(z) = Z[x(n)] \quad (4.3)$$

In Eq.(4.1) the summation is taken from $n = 0$ to ∞ , ie, $X(z)$ is not at all related to the past history of $x(n)$. This is one-sided or unilateral z-transform. Sometime the one-sided z-transform has to take into account the initial conditions of $x(n)$ (see section 4.7).

In general, signals exist at all time, and the two-sided or bilateral z-transform is defined as

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} \\ &= x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \dots \end{aligned} \quad (4.4)$$

Because $X(z)$ is an infinite power series of z^{-1} , the transform only exists at values where the series converges (i.e. goes to zero as $n \rightarrow \infty$ or $-\infty$). Thus the z-transform is accompanied with its region of convergence (ROC) where it is finite (see section 4.4).

A number of authors denote $X^+(z)$ for one-side z-transform.

Example 4.1.1

Find the z-transform of the two signals of Fig.4.1

Solution

(a) Notice the signal is causal and monotonically decreasing and its value is just 0.8^n for $n \geq 0$. So we write

$$x(n) = 0.8^n u(n)$$

and use the transform (4.1)

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(n)z^{-n} \\ &= 1 + 0.8z^{-1} + 0.64z^{-2} + 0.512z^{-3} + \dots \\ &= 1 + (0.8z^{-1}) + (0.8z^{-1})^2 + (0.8z^{-1})^3 + \dots \end{aligned}$$

Applying the formula of infinite geometric series which is repeated here

$$1 + a + a^2 + a^3 + \dots = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \quad |a| < 1 \quad (4.5)$$

to obtain

$$X(z) = \frac{1}{1-0.8z^{-1}} = \frac{z}{z-0.8}$$

The result can be left in either of the two forms .

(b) The signal is alternatively positive and negative with increasing value .The signal is divergent . We can put the signal in the form

$$x(n) = (-1.2)^{n-1} u(n-1)$$

which is $(-1.2)^n u(n)$ delayed one index(sample) . Let's use the transform (4.1)

$$\begin{aligned} X(z) &= \sum_{n=0}^{\infty} x(n)z^{-n} \\ &= 0 + 1.0(z^{-1}) - 1.2(z^{-1})^2 + 1.44(z^{-1})^3 - 1.718(z^{-1})^4 + \dots \\ &= z^{-1}[1 + (-1.2z^{-1}) + (-1.2z^{-1})^2 + (-1.2z^{-1})^3 + \dots] \end{aligned}$$

$$= z^{-1} \frac{1}{1+1.2z^{-1}} = \frac{z^{-1}}{1+1.2z^{-1}} = \frac{1}{z+1.2}$$

1.2 The inverse z-transform

The inverse z-transform is denoted by Z^{-1} :

$$x(n) = Z^{-1}[X(z)] \quad (4.6)$$

The signal $x(n)$ and its transform constitutes a transform pair

$$X(n) \leftrightarrow Z(z) \quad (4.7)$$

One way to find the inverse transform , whenever possible , is to utilize just the z-transform definition. General methods of the inverse z-transform are discussed in section 4.5 and 4.6

Example 4.1.2

Find the inverse z-transform of the following

1. $X(z) = \frac{z}{z-0.8}$
2. $\frac{1}{z+1.2}$

Solution

(a) Let's write

$$\begin{aligned} X(z) &= \frac{z}{z-0.8} = \frac{1}{1-0.8z^{-1}} \\ &= 1 + (0.8z^{-1}) + (0.8z^{-1})^2 + (0.8z^{-1})^3 + \dots \\ &= 1 + 0.8z^{-1} + 0.64z^{-2} + 0.512z^{-3} + \dots \end{aligned}$$

By comparing term by term with Equation (4.2) we get

$$x(n) = [1, 0.8, 0.64, 0.512, \dots]$$

or

$$x(n) = 0.8^n u(n)$$

(b) Let's write

$$X(z) = \frac{1}{z+1.2} = \frac{z^{-1}}{1+1.2z^{-1}} = z^{-1} \frac{1}{1+1.2z^{-1}}$$

Next , let's expand $X(z)$:

$$\begin{aligned} X(z) &= z^{-1}[1 + (-1.2z^{-1}) + (-1.2z^{-1})^2 + (-1.2z^{-1})^3 + \dots] \\ &= 0 + 1.0z^{-1} - 1.2z^{-2} + 1.44z^{-3} - 1.728z^{-4} + \dots \end{aligned}$$

Thus

$$x(n) = [0, 1.0, -1.2, 1.44, -1.728, \dots]$$

or

$$x(n) = (-1.2)^{n-1} u(n-1)$$

That is

$$\begin{aligned} x(n) &= 0 \quad n \leq 0 \\ &= (-1.2)^n \quad n > 0 \end{aligned}$$

1.3 z-transform pairs

Table 4.1 gives many useful z-transform pairs . All signals are causal (right-sided), except two which are anticausal (left-sided). Notice that a transform can be expressed equivalently as a function of z^{-1} or z , for example

Table 4.1 : Common z-transform pairs

$$u(n) \rightarrow X(z) = \frac{1}{1-z^{-1}} \text{ or } \frac{z}{z-1}$$

$$a^n u(n) \rightarrow X(z) = \frac{1}{1-az^{-1}} \text{ or } \frac{z}{z-a}$$

$$(\cos n\Omega_0)u(n) \rightarrow X(z) = \frac{1-z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}} \text{ or } \frac{z(z-\cos\Omega_0)}{z^2-2z\cos\Omega_0+1}$$

Which form is more appropriate depending on what we would like to do with the transform. (see sections 4.1.6 , 4.3 and 4.6).

1.4 z–transform for systems

The z-transform applies to systems as well as signals because systems are represented by their impulse responses which are functions of index n (time or space ...) just like signals. Remember that many other transforms (Laplace, Fourier ...) have the same property, and due to this property that the transforms are useful in the analysis and design of systems because signals and systems interact.

Specifically , the z–transform of impulse response h(n) is

$$H(z) = \sum_{n=0}^{\infty} h(n)z^{-n} \text{ (one–sided transform) (4.8)}$$

or

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} \text{ (two–sided transform) (4.9)}$$

depending on whether the system is causal or noncausal .

$H(z)$, the z–transform of $h(n)$, is called transfer function or system function of the system .

Example 4.1.3

A system has impulse response

$$h(n) = [1, 2, 3, 4, 5, 6]$$

Find the transfer function .

Solution

The system is of noncausal FIR type . Its transfer function is

$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=-2}^3 h(n)z^{-n} \\ &= h(-2)z^2 + h(-1)z^{-1} + h(0)z^0 + h(1)z^{-1} + h(2)z^{-2} + h(3)z^{-3} \\ &= z^{-2} + 2z^{-1} + 3 + 4z^{-1} + 5z^{-2} + 6z^{-3} \end{aligned}$$

On the contrary, if $H(z)$ is known as above we can easily obtain $h(n)$.

1.5 Eigen-function and eigen-value

We know that if the frequency response of a system is $H(\omega)$ then for input $x(n) = e^{jn\omega}$, the output is $y(n) = e^{jn\omega} H(\omega)$. Because of this , $e^{jn\omega}$ is the eigen-function , and $H(\omega)$ the eigen-value of the system.

Now , for input

$$x(n) = z^n \quad (4.10)$$

the system output is

$$y(n) = h(n) * x(n) = \sum_{k=0}^{\infty} h(k)z^{n-k} = z^n \left[\sum_{k=0}^{\infty} h(k)z^{-k} \right]$$

In the brackets is just $H(z)$, thus

$$y(n) = z^n H(z) \quad (4.11)$$

Hence in the z–transform domain , z^n is eigen-function and $H(z)$ is eigen-value of the system.

1.6 Transfer function in terms of filter coefficients

For , Let's begin with the general filter difference equation which is repeated here

$$y(n) = \sum_{k=1}^M a_k y(n-k) + \sum_{k=-N}^N b_k x(n-k) \quad (4.12)$$

where a_k and b_k are the filter coefficients (constants) , and the limits M, N can be extended to infinity .

Now we make the replacement $x(n) = z^n$ and $y(n) = z^n H(z)$ to get

$$z^n H(z) = \sum_{k=1}^M a_k z^{n-k} H(z) + \sum_{k=-N}^N b_k z^{n-k}$$

From this we derive the expression of $H(z)$

$$H(z) = \frac{\sum_{k=-N}^N b_k z^{-k}}{1 - \sum_{k=1}^M a_k z^{-k}} \quad (4.13)$$

It is worthwhile to notice that the above transfer function is resulted from the filter equation (4.12). Other authors write the filter equation differently (for example, all the y terms are on the left side of the equation), leading to a slightly different expression of $H(z)$. For nonrecursive filters the denominator is just 1 and $H(z)$ becomes that of nonrecursive filters as we know.

The idea here is that when the filter equation is given, we collect its coefficients to place into the expression above of $H(z)$ without the need to take the z -transform. Vice versa, if we know $H(z)$ that means we know the filter coefficients, hence the filter equation.

Example 4.1.4

Given

$$(a) H(z) = \frac{2z^2 - 3z}{z^2 + 0.5z - 0.8}$$

$$(b) H(z) = \frac{-20z^2 + 5z}{10z^3 + 5z^2 - 8z - 1}$$

Find the filter difference equation.

Solution

(a) Write $H(z)$ as a function of z^{-1} by multiplying the numerator and denominator with z^{-2} :

$$H(z) = \frac{2 - 3z^{-1}}{1 + 0.5z^{-1} - 0.8z^{-2}} = \frac{2 - 3z^{-1}}{1 - (-0.5z^{-1} + 0.8z^{-2})}$$

The coefficients are

$$a_0 = 2 \quad a_1 = -3 \quad b_1 = -0.5 \quad b_2 = 0.8$$

Thus the filter equation is

$$y(n) = -0.5y(n-1) - 0.8y(n-2) + 2x(n) - 3x(n-1)$$

(b) Multiply the numerator and denominator with $0.1z^{-3}$ to make $10z^{-3}$ in the denominator equal to 1 :

$$H(z) = \frac{-2z^{-1} + 5z^{-2}}{1 + 0.5z^{-1} - 0.8z^{-2} - 0.1z^{-3}}$$

Collect the coefficients:

$$a_1 = -2 \quad a_2 = 5 \quad b_1 = -0.5 \quad b_2 = 0.8 \quad b_3 = -0.1$$

Thus

$$y(n) = -0.5y(n-1) + 0.8y(n-2) + 0.1y(n-3) - 2x(n-1) + 5x(n-2)$$

$y(n)$